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A STUDY OF ONE-DIMENSIONAL ICE FORMATION WITH PARTICULAR REFERENCE TO PERIODIC GROWTH AND DECAY

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Abstract—The paper considers one-dimensional ice formation near the surface of a semi-infinite domain. A series of experiments executed with water are described and the results compared with theoretical predictions for power law and sinusoidal variations in surface temperature. The theoretical study is divided into two parts—analytic and numerical. The former consists of approximate solutions developed from a perturbation expansion and the latter involves discretization of the space variables and the integration of the resulting set of first order non-linear equations.

NOMENCLATURE

ϕ, θ,	temperature departure from transition temperature;
τ, t,	time;
x, X,	depth from stationary upper surface;
β, b,	depth of interface;
Y,	X/b;
k,	thermal conductivity;
ρ,	density;
c_p,	constant pressure specific heat;
κ,	thermal diffusivity;
L,	latent heat of fusion;
Ste,	Stefan number;
F,	function defined in text.

Subscripts

i,	initial;
c,	characteristic;
β, b,	interface;
s,	solid;
L,	liquid.

Superscript

0,	stationary upper surface.

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INTRODUCTION

THE STUDY of heat conduction with change of phase began [1] and has continued with problems of ice formation. These problems and the related studies of frost penetration are of considerable interest in Canada, the United States and Russia which are partly situated in polar and sub-polar regions. During the latter half of the last century a considerable effort has been expended towards predicting the growth and decay characteristics of ice and consequently our understanding the process has improved substantially. Despite this, predictions of the depth of ice forming at the surface of a semi-infinite domain under given annual meteorological conditions appear to be unavailable.

Neumann's classic solution [1] for a step change in surface temperature is probably the most commonly used analytic result today, despite its departure from observed variations in surface temperature and notwithstanding the the many extensions and improvements incorporated in recent analytic work [2–10]. None of these papers discusses periodic boundary conditions and it is uncertain whether the most general techniques suggested would be capable of handling them without modification.

Numerical methods [11–14] have received some attention and appear to offer greater flexibility, presumably because of their relative insensitivity to variations in the boundary conditions. Analogue methods have also been considered [15–18]. Very few attempts have been made to apply numerical or analogue methods to the periodic problem and as pointed out by Scott [19] the attempts have been restricted to individual cases.

Aggravating the limitations of the analytic and numerical predictions is the scarcity of detailed experimental results. Field data are widely scattered throughout the literature [20–22] but usually suffer from the drawback of uncertainty in the conditions and material properties. Laboratory experiments, on the other hand, are sparsely reported and very limited in the type of test and range of conditions covered [23–26].

The work presented here is an attempt to develop numerical and approximate analytic solutions to the plane ice problem, incorporating the periodic boundary condition in particular, and to compare these solutions with each other and with a set of laboratory experiments.

APPROXIMATE ANALYTIC METHOD

Formulation

Consider ice formation in the one-dimensional domain described by Fig. 1. Heat conduction within the ice is governed by the equation

$$\frac{\partial^2\theta}{\partial X^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}. \tag{1}$$

At the ice–water interface a heat balance yields the following:

$$\rho L\frac{\mathrm{d}b}{\mathrm{d}t} = k_s\left(\frac{\partial\theta_s}{\partial X}\right)_b - k_L\left(\frac{\xi\theta_L}{\partial X}\right)_b \tag{2}$$

where the subscripts s, L denote the solid and liquid phases respectively.

Normalizing the last equation, and ignoring

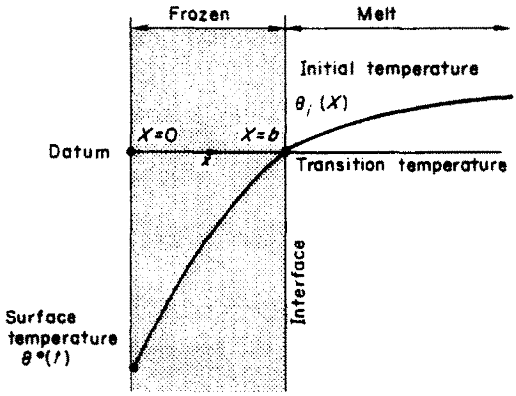

FIG. 1. Co-ordinate system.

the temperature distribution in the liquid phase for simplicity, we obtain

$$\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = \left(\frac{\partial\phi}{\partial x}\right)_\beta \tag{3}$$

where

$$\tau = \frac{t}{t_c}, \qquad x = \left(\frac{\rho L}{k\theta_c t_c}\right)^{\frac{1}{2}} X$$

$$\phi = \frac{\theta}{\theta_c}, \qquad \beta = \left(\frac{\rho L}{k\theta_c t_c}\right)^{\frac{1}{2}} b$$

and the subscript c refers to a reference quantity which provides a measure of the scale of the corresponding variable. Substitution of the above normalized variables into equation (1) then yields

$$\frac{\partial^2\phi}{\partial X^2} = Ste\,\frac{\partial\phi}{\partial\tau} \tag{4}$$

where $Ste = (c_p\theta_c/L)$ is a quantity indicating the relative importance of sensible and latent heat effects; it will be referred to as the Stefan number.

The progress of the ice–water interface is determined from the solution of equation (3) subject to a suitable initial condition; for example $\beta = 0$, if the system is initially ice-free. The solution of equation (3) in turn depends upon the solution of equation (4) which must also satisfy appropriate initial and boundary conditions. In this paper these conditions will be defined through a prescribed variation of surface temperature and a liquid temperature always taken as the equilibrium freezing temperature: it is thus implied that $\theta_f(X) = 0$.

Typically, $Ste \ll 1$ for an ice–water system and this suggests [27] a solution of equation (4) in the form

$$\phi(x, \tau, Ste) = \phi_0(x, \tau) + \sum_{p=1}^{\infty} Ste^p \, \phi_p(x, \tau) \quad (5)$$

Substituting this into equation (4) generates the following set of equations:

$$\begin{aligned} &\frac{\partial^2 \phi_0}{\partial x^2} = 0 \\ &\frac{\partial^2 \phi_p}{\partial x^2} = \frac{\partial \phi_{p-1}}{\partial \tau} (p = 1, 2, \ldots \infty) \end{aligned} \quad (6)$$

the solutions of which must satisfy the requirements that

$$\begin{aligned} \phi_0(0, \tau) &= \phi^0(\tau) \\ \phi_0(\beta, \tau) &= 0 \\ \phi_p(0, \tau) &= 0 \\ \phi_p(\beta, \tau) &= 0. \end{aligned} \quad (7)$$

From the solutions equations (6) it is found that the temperature field is described by

$$\phi(x, \tau, Ste) = a_1(\tau)\, x + a_2(\tau) + Ste\left[\frac{x^3}{3!}\frac{da_1}{d\tau} + \frac{x^2}{2!}\frac{da_2}{d\tau} + a_3(\tau)\, x + a_4(\tau)\right] + O(Ste^2).$$

Satisfaction of equations (7) then yields

$$\phi(x, \tau, Ste) = \phi^0(\tau)\left(1 - \frac{x}{\beta}\right) + Ste\, \phi^0(\tau)\left[-\left(\frac{1}{3!}\frac{d\beta}{d\tau} + \frac{\beta}{3\phi^0}\frac{d\phi^0}{d\tau}\right)x + \left(\frac{1}{\phi^0}\frac{d\phi^0}{d\tau}\right)\frac{x^2}{2!} + \left(\frac{1}{\beta^2}\frac{d\beta}{d\tau} - \frac{1}{\beta\phi^0}\frac{d\phi^0}{d\tau}\right)\frac{x^3}{3!}\right] + \ldots \quad (8)$$

Similarly, by taking

$$\beta(\tau, Ste) = \beta_0(\tau, Ste) + \sum_{p=1}^{\infty} Ste^p\, \beta_p(\tau, Ste) \quad (9)$$

equation (3) generates

$$\begin{aligned} \frac{d\beta_0}{d\tau} &= \left(\frac{\partial \phi_0}{\partial x}\right)_\beta \\ \frac{d\beta_p}{d\tau} &= \left(\frac{\partial \phi_p}{\partial x}\right)_\beta . \end{aligned} \quad (10)$$

The use of equation (8) in the first of equations (10) thus gives

$$\frac{d\beta_0}{d\tau} = -\frac{\phi^0(\tau)}{\beta_0 + Ste\, \beta_1 + O(Ste^2)}$$

which integrates formally to give

$$\beta_0(\tau, Ste) = F_0(\tau) + \frac{Ste}{F_0(\tau)}\int_0^\tau \frac{\beta_1(\tau, Ste)\, \phi^0(\tau)\, d\tau}{F_0(\tau)}, \quad (11)$$

where

$$F_0(\tau) = \{-2\int_0^\tau \phi^0(\tau)\, d\tau\}^{\frac{1}{2}}$$

and it has been assumed that $\beta_0(0, Ste) = 0$. At this point, and from now on, terms of higher order in Ste are neglected. The second of equations (10) may be solved in a similar manner subject to $\beta_1(0, Ste) = 0$. The equation first reduces to

$$\frac{d\beta_1}{d\tau} = \frac{\phi^0(\tau)}{6}\left[\frac{\beta_0(\tau)}{\phi^0(\tau)}\frac{d\phi^0}{d\tau} - 2\frac{\phi^0(\tau)}{\beta_0(\tau)}\right]$$

which may be integrated formally to yield

$$\beta_1(\tau, Ste) = \frac{1}{6}\int_0^\tau \beta_0(\tau)\frac{d\phi^2}{d\tau}\, d\tau - \frac{1}{3}\int_0^\tau \frac{\{\phi^0(\tau)\}^2}{\beta_0(\tau)}\, d\tau.$$

Following substitution of the zeroth-order term for β_0 from equation (11) [since this represents a first-order contribution within $\beta_1(\tau, Ste)$] the equation immediately above then assumes the form

$$\beta_1(\tau, Ste) = \frac{1}{6}\int_0^\tau F_0(\tau)\frac{\mathrm{d}\phi^0}{\mathrm{d}\tau}\,\mathrm{d}\tau - \frac{1}{3}\int_0^\tau \frac{\{\phi^0(\tau)\}^2}{F_0(\tau)}\,\mathrm{d}\tau. \tag{12}$$

Equation (12) is an explicit form which, for any prescribed surface temperature, yields part of the first-order contribution to $\beta(\tau, Ste)$. The other part may be calculated by substitution of equation (12) into the second term on the right-hand side of equation (11). Thus equations (9), (11) and (12) readily provide a first-order approximation for ice thickness in an explicit form.

Solutions

At first sight, the above formulation appears to be valid for an arbitrary variation in surface temperature but closer examination reveals that this is not true. If the sign of $\phi^0(\tau)$ were to change at any time it is clear that a freezing process would be converted into a melting process, or vice versa. Since the analysis given does not strictly accommodate the melting and freezing of such strata it must be restricted to situations in which $\phi^0(\tau)$ is unchanging in sign. No other restriction need be applied to the form of the surface temperature.

From the wide range of possibilities the form of special interest here is the sinusoidal surface temperature, which is obviously not constant in sign. To circumvent this difficulty solutions will be calculated piecewise, each piece corresponding to the layer closest to the surface during a surface temperature excursion on only one side of the freezing point. As mentioned above, this procedure ignores the dynamic effect of any other ice or water layers, but the error so incurred is not as large as might at first appear. This is because $Ste \ll 1$ and therefore each layer will rapidly assume and remain at the freezing temperature until such time as the surface layer contacts it once more. Beyond that time the magnitude of the Stefan number again ensures that any additional sensible heat exchange will not greatly alter the temperature régime.

The formation of ice from water will now be considered by taking

$$\phi^0(\tau) = -\sin\tau \tag{13}$$

with $0 \leqslant \tau \leqslant \pi$. It follows immediately from equation (8) that the temperature distribution in the ice is then given by

$$\begin{aligned}\phi(x, \tau, Ste) = &-\left(1 - \frac{x}{\beta}\right)\sin\tau \\ &- Ste\sin\tau\left[-\left(\frac{3}{3!}\frac{\mathrm{d}\beta}{\mathrm{d}\tau} + \frac{\beta\cos\tau}{3\sin\tau}\right)x\right. \\ &\left.+\left(\frac{\cos\tau}{\sin\tau}\right)\frac{x^2}{2!} + \left(\frac{1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}\tau} - \frac{\cos\tau}{\beta\sin\tau}\right)\frac{x^3}{3!}\right].\end{aligned} \tag{14}$$

Since

$$\phi^0(\tau) = -\sin\tau, F_0(\tau) = 2\sin\frac{\tau}{2},$$

and therefore from equation (12),

$$\beta_1(\tau, Ste) = -\frac{2}{9} - \frac{2}{3}\cos\frac{\tau}{2} + \frac{8}{9}\cos^3\frac{\tau}{2}.$$

Substituting this into equation (11) and then into equation (9) we obtain, after some algebra,

$$\beta(\tau, Ste) = 2\sin\frac{\tau}{2} - \frac{Ste}{3}\sin\frac{\tau}{2}\sin\tau. \tag{15}$$

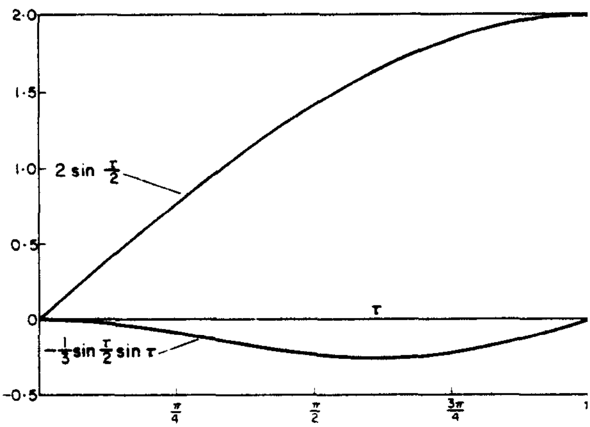

FIG. 2. Zeroth and first-order terms.

Both zeroth and first-order terms in equation (15) are simple functions and are plotted in Fig. 2 merely for the sake of clarity. The figure reveals that the first-order term is about an order-of-magnitude less than the zeroth-order term thus emphasizing the accuracy of the latter. The figure, and equation (15), also reveals the curious fact that the overall depth of freezing (or thawing) is unaltered by the inclusion of the first-order term. Therefore the extent of the active zone is given by $\beta = 2$ to within an accuracy of order Ste^2.

A similar procedure may be used for power law variations in the departure of the surface temperature from the freezing temperature. The corresponding expressions for temperature and interface depth are

$$\phi(x,\tau) = -\left(1 - \frac{x}{\beta}\right)\tau^n - Ste\,\tau^n\left[-\left(\frac{1}{3!}\frac{\mathrm{d}\beta}{\mathrm{d}\tau} + \frac{\beta n}{3\tau}\right)x + \left(\frac{n}{\tau}\right)\frac{x^2}{2!} + \left(\frac{1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}\tau} - \frac{n}{\beta\tau}\right)\frac{x^3}{3!}\right]$$

$$\beta(\tau, Ste) = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\tau^{(n+1)/2} - \frac{Ste}{6}\left(\frac{2}{n+1}\right)^{\frac{1}{2}}\tau^{(3n+1)/2},$$

respectively.

NUMERICAL METHOD

Formulation

It is convenient to formulate the numerical problem in a moving coordinate system. Defining $y = X/b$, equations (1) and (2) may be rewritten as

$$\frac{\partial\theta}{\partial t} = \frac{y}{b}\frac{\mathrm{d}b}{\mathrm{d}t}\frac{\partial\theta}{\partial y} + \frac{\kappa}{b^2}\frac{\partial^2\theta}{\partial y^2} \tag{16}$$

and

$$\rho L b\frac{\mathrm{d}b}{\mathrm{d}t} = k_s\left(\frac{\partial\theta_s}{\partial y}\right)_{y=1} - k_L\left(\frac{\partial\theta_L}{\partial y}\right)_{y=1}, \tag{17}$$

which clearly reveal the non-linearity associated with the moving interface. Appropriate initial and boundary conditions are:

$$\theta(y,0) = \theta_i(y)$$
$$\theta(0,t) = \theta^0(t)$$
$$\theta(1,t) = 0 \quad \text{and} \quad b(0) = b^0.$$

The spatial derivatives are now replaced by the approximate difference relations

$$\left(\frac{\partial^2\theta}{\partial y^2}\right)_{y=(j/m)} \simeq \frac{m}{2}(\theta_{j+1} - 2\theta_j + \theta_{j-1})$$
$$\left(\frac{\partial\theta}{\partial y}\right)_{y=(j/m)} \simeq \frac{m}{2}(\theta_{j+1} - \theta_{j-1})$$

and

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=1} \simeq \frac{m}{6}(11\theta_m - 18\theta_{m-1} + 9\theta_{m-2} - 2\theta_{m-3}),$$

by supposing y to be subdivided into m (six per layer) equal parts. Substituting the above discretized forms into equations (16) and (17) yields a set of m simultaneous first-order, non-linear differential equations. Thus,

$$\begin{aligned}\frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= g_1(\theta^0, \theta_1 \ldots \theta_m, b)\\ &\vdots\\ \frac{\mathrm{d}\theta_{m-1}}{\mathrm{d}t} &= g_{m-1}(\theta^0, \theta_1, \ldots \theta_m, b)\end{aligned} \tag{18}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = g_m(\theta^0, \theta_1 \ldots \theta_m, b) \tag{19}$$

where θ^0 and θ_m are prescribed by the boundary conditions. It is now apparent that arbitrary variations in the surface temperature $\theta^0(t)$ present no special difficulty, provided the freezing temperature is not crossed.

Integration

Equations (18) and (19) pose an initial value problem which, apart from the approximations implicit in truncation, is a complete and accurate description of the ice–water system. Several numerical techniques are available for their solution and from these the fourth order Runge–Kutta procedure was chosen. This method proved satisfactory except when the ice depth b became very small, where it required extremely small time increments for the maintenance of stability. It is known [29] that for the diffusion equation within fixed boundaries the stability requirement is $[\kappa\Delta t/(\Delta X)^2] \leqslant 0{\cdot}7$. The corresponding requirement for moving boundaries is unknown but it was found that with $(m^2\kappa\Delta t/b^2) \leqslant 0{\cdot}5$ stability was maintained.

Numerical duplication of the analytic and experimental results for monotonic boundary conditions was not undertaken but there is every indication that no special difficulties would have arisen. In the periodic problem there was only one major difficulty which has already been mentioned. During formation of a new layer or vanishing of an old one the thickness must necessarily tend to zero, with attendant loss of stability. This difficulty was circumvented satisfactorily by assuming a linear temperature profile during periods of "formation" or "pinch-out" and using equation (19) as before. Despite this, the fraction of the cycle during which layers were small was found to be great enough that excessive computational times were encountered. Using the IBM 7040 machine required 1 h to execute three 2 h cycles.

EXPERIMENTAL STUDY

Equipment and instrumentation

The apparatus is shown diagramatically in Fig. 3. It consisted [28] of a transparent-walled, rectangular plastic tank ($11 \times 8 \times 7{\cdot}5$ in. approx.) cooled at the top by means of a vapour compression refrigerator. The sides and base were insulated by thick layers of styrofoam, a "window" of which, situated on the front side, could be removed for periodic inspection of the ice growth. Silicone rubber cushions were inserted on each of two sides to reduce the danger of expansion breakage.

The temperature of the refrigerant passing through the apparatus was conveniently controlled by adjusting the pressure-regulating valve in the return line in all but the sinusoidal tests, for which temperatures in excess of the freezing temperature were adjusted with resistive electric heaters. Accuracy was about ± 2°F in the periodic tests and better than ± 1°F in all others. Each time-dependent boundary condition was obtained by carefully approximating the desired function with a series of small step changes.

Throughout each run the progress of ice-water interfaces was followed by frequent periodic observation. Each interface was found to be a sharply defined plane and its depth was measured by alignment of the plane with a pair of sights, the location of which was determined externally to within $\pm 0{\cdot}002$ in. With this arrangement the estimated uncertainty in the measured ice depths was $\pm 0{\cdot}005$ in. The observations revealed that distortion at the edge of the interface was negligible, thus indicating that the experiments could have been executed with a much smaller volume of water.

Test procedure

In several preliminary test runs it was found that ice formation did not occur immediately after lowering the surface temperature beneath 32°F. This indicated sub-cooling of the water, a fact which, in view of the use of unprepared tap water, came as a surprise. The effect became very significant in certain tests and in order to eliminate it from the proposed tests the following procedure was adopted. After cooling the bulk of the water to 32°F the surface temperature was lowered in a convenient manner until water crystallized at the surface. The crystallization produced a finite layer of ice which was then thawed back (with the surface fixed near 32°F) until it was about to vanish. At this point the test proper was started and the surface temperature then lowered in the desired manner, beginning at 32°F.

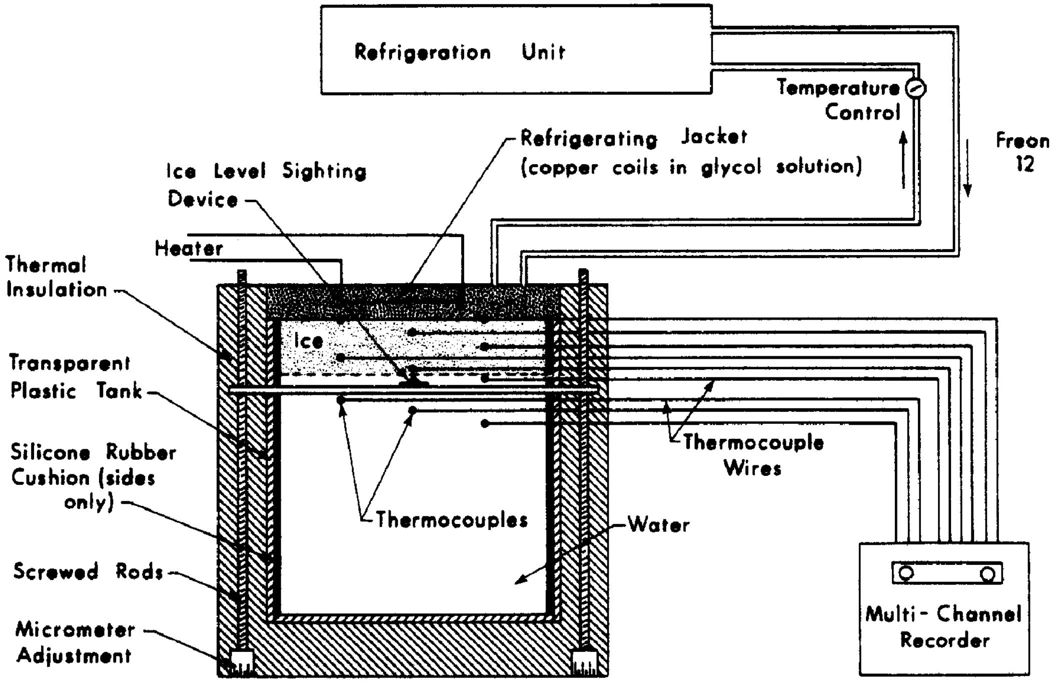

FIG. 3. Schematic of apparatus.

DISCUSSION AND CONCLUSIONS

Monotonic interface history

The measured progress of the interface for two power law variations is shown in Fig. 4. Also plotted are the results of the approximate analysis for each of the prescribed experimental forms, plus the Neumann solution. For the particular powers prescribed the results are seen to be closely grouped and indicate good agreement between theory and experiment: the discrepancy in the least accurate test ($n = 0{\cdot}83$) is attributed to the difficulty in choosing a temporal origin.

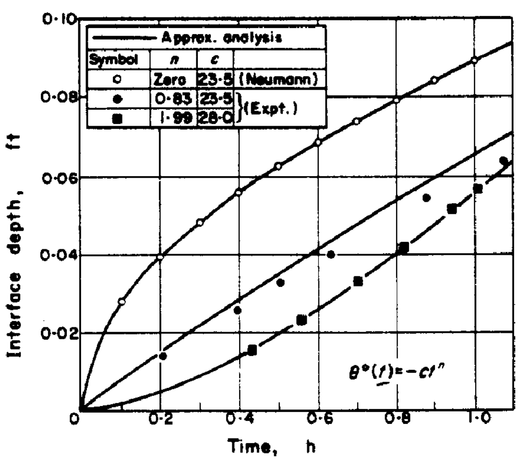

FIG. 4. Power law interface histories.

It is unfortunate that measured values were not obtained for powers close to $n = 0$ but it was found that a step discontinuity in surface temperature was very difficult to approach experimentally. Nevertheless, the exact and approximate curves in Fig. 4 clearly indicate that boundary conditions similar to, but not identical with, the Neumann condition lead to large departures from the predictions of the Neumann solution.

Periodic interface history

Figure 5 combines the analytic, numerical and experimental results for periodic freezing and thawing over the first three cycles. During the first half-cycle a thick layer of ice formed and with each successive freezing half-cycle the lower ice face penetrated further into the water [28]. Results for this interface have been omitted from the figure for simplicity and because of uncertainty in the experimental conditions.

In keeping with the earlier discussion the analytic results have been plotted piecewise over τ intervals of π: ice properties were used during a freezing half-cycle and water properties during thawing. It is evident from the figure that agreement between the analytic and numerical results is excellent. The neglect of the dynamic effects of regions outside the layer adjacent to the stationary upper surface is seen to be small, as the magnitude of the Stefan number (0·061) would imply.

It is also evident from Fig. 5 that the theoretical melting predictions are well short of the measured values, particularly during the later stages. Bearing in mind the uncertainty in the properties of tap water, frozen or unfrozen, and the neglect of natural convection it would appear that closer agreement could not be expected. A very recent paper [26] published after the time the experiments reported here were executed contains data which suggest that natural convection did influence the present experiments. These more recent data, although obtained for a step change in surface temperature, imply that the effect would be most noticeable in the later stages of the melting process and would produce a greater thaw depth. The figure reveals such behaviour with an increase of about 16 per cent in the depth of thaw.

A recurrent phenomenon revealed by Fig. 5 is the delay evident in the early stages of each melting period and not present in the corresponding freezing periods (neglecting the first freezing period for other reasons). Clearly, this cannot be a subcooling effect and the explanation appears to lie with two other possibilities. Firstly, in passing through 32°F the apparatus controlling surface temperature had to be switched: from a refrigerator to a heater or vice versa. Since the effective thermal capacity of the cooling system was much greater than that of the heating system it is likely that the ascent through 32°F entailed greater delay than the descent. Secondly, it is possible that the greater density of melted ice created a thin vapour film separa-

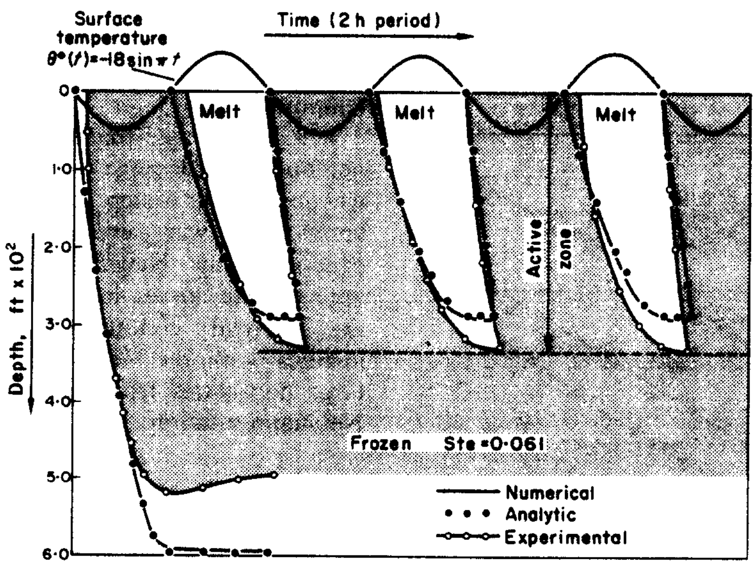

FIG. 5. Periodic interface history.

ting the water and the heated surface. No such film was noticed although it may have been thin enough and uniform enough to escape attention.

Conclusions

The laboratory experiments were originally designed to test the analytic and numerical predictions with a view to their future use in field conditions. In this way it was hoped that a precisely posed theoretical problem could be simulated experimentally and the validity of the assumptions checked within the accuracy of the experiments.

The power law tests were well-suited to the characteristics of the apparatus and incorporated the full accuracy of the equipment. Since the water was initially cooled to 32°F the effect of convection was negligible. Figure 4 indicates that under these conditions the approximate analysis provides simple, reliable solutions which will be valid in similar circumstances, e.g. late fall and early winter, especially in a lake or newly exposed ground areas such as occur after surface stripping and/or backfill.

In the periodic tests very accurate reproduction of the prescribed theoretical conditions was not possible. This has been attributed to departures from a perfect sinusoidal variation in the boundary temperature and the effect of natural convection, particularly during the period of thaw. Like subcooling, natural convection may have a significant effect in the field (e.g. lake and sea ice) and the laboratory.

Round-off and truncation error in the numerical results is considered to be quite small and therefore it is suggested that the numerical results provide an accurate description of periodic behaviour in the absence of convection. The analytic predictions for interface depth are in close agreement with the numerical results and are about 16 per cent less than the measured values. The simplicity of the analytic expression makes it useful for most field situations but it must be noted that the analysis was restricted to a constant-property, homogeneous medium of high moisture content. The numerical technique, on the other hand, does not suffer from these limitations providing, of course, that the property variations are known. Possibly, the excessive computational times encountered could be reduced by a normalization process.

One of the most important features of the sinusoidal results was the agreement between theory and experiment in revealing the rapidity with which steady periodic behaviour is approached. The active zone, which is effectively defined within the first cycle, typifies this situation. The fact that transients played a negligible role is implied by the magnitude of the Stefan number. It is therefore concluded that when $Ste \ll 1$ the interface depth is much more strongly influenced by immediate history than by the remote past.

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ETUDE DE LA FORMATION UNIDIMENSIONNELLE DE LA GLACE AVEC MENTION PARTICULIÈRE DE LA CROISSANCE ET DE LA DÉCROISSANCE PÉRIODIQUES

Résumé—L'article considère la formation unidimensionnelle de glace, près de la surface d'un domaine semi-infini. Une série d'expériences exécutées avec de l'eau est décrite et les résultats sont comparés avec les prévisions théoriques pour des variations en puissance et sinusoïdales de la température de surface. L'étude théorique est divisée en deux parties—analytique et numérique. La première consiste en solutions approchées élaborées à partir d'un développement de perturbations et la dernière implique une discrétisation des variables d'espace et l'intégration du système d'équations nonlinéaires du premier ordre qui en résulte.

EINE UNTERSUCHUNG DER EIN-DIMENSIONALEN EISBILDUNG MIT BESONDERER BERÜCKSICHTIGUNG PERIODISCHEN ANWACHSENS UND ZERFALLS

Zusammenfassung—Es wird die ein-dimensionale Eisbildung an der Oberfläche eines halb-unendlichen Bereiches betrachtet. Eine Reihe von Versuchen mit Wasser werden beschrieben und die Ergebnisse mit theoretischen Berechnungen verglichen, die ein Potenzgesetz und sinusförmige Verteilungen der Oberflächentemperatur vorschreiben. Die theoretische Untersuchung ist in zwei Teile, einen analytischen und einen numerischen geteilt. Der erstere besteht aus Näherungs-lösungen die aus einer Störfunktion folgen und der letztere umfasst die Aufteilung von Raumveräanderlichen und die Integration des sich ergebenden Satzes nicht-linearer Differentialgleichungen erster Ordnung.

Аннотация—В статье рассматривается одномерное образование льда вблизи поверхности полуограниченной области. Описываются эксперименты с водой и полученные результаты сравниваются с теоретическими расчетами для случаев степенного и синусоидального изменения температуры поверхности. Теоретическое исследование состоит из двух частей: анализа и расчета. Первая часть состоит из приближенных решений, полученных для случая неупорядоченного увеличения объема льда. Во второй части проводится разделение пространственных переменных и интегрирование полученной системы нелинейных уравнений первого порядка.